Metric Space

Def^n: Let E be ~~m~~ any non-empty set. Then a mapping
$$d : E \times E \to R^+$$
is called metric over E if

m_1 (i) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

m_2 (ii) $d(x,y) = d(y,x)$ (Symmetric property)

m_3 (iii) $d(x,y) \leq d(x,z) + d(z,y)$

Law of triangle inequality

$\forall\, x, y, z \in E$

Then the pair (E, d) is a metric space.

e.g. (1) Let E be a non-empty set.

Let $d : E \times E \to R$ be defined as
$$d(x,y) = 1 \text{ if } x \neq y$$
$$= 0 \text{ if } x = y$$
Then show that d is a metric over E.

Verification:-

To verify m_1, we have
$d(x,y) \geq 0$ (given)
$\therefore d(x,y) = 0$ iff $x = y$ (given)
Hence m_1 is satisfied.
Obviously $d(x,y) = d(y,x)$
Hence m_2 is satisfied
To verify m_3. Let $x, y, z \in E$ be arbitrary.
If $x = y$
$$d(x,y) \leq d(x,z) + d(z,y)$$
is obviously satisfied.

if $x \neq y$, then at least one of $x \neq z$ & $y \neq z$

So,

$d(x,z) + d(z,y) \geq 1 = d(x,y)$

i.e., $d(x,y) \leq d(x,z) + d(z,y)$

Therefore in both cases m_3 holds

Hence (E,d) is metric space

e.g (2) Let R be the set of real numbers.

Let $d: R \times R \to R$

be defined as $d(x,y) = |x-y|$
$\forall x,y \in R$

Proof:- For m_1

$d(x,y) = |x-y| \geq 0$

& $d(x,y) = 0 \Rightarrow |x-y| = 0 \Rightarrow x = y$

Hence m_1 holds

For m_2

$d(x,y) = |x-y| = |y-x| = d(y,x)$

Hence m_2 satisfied.

For m_3

Let us take $x,y,z \in R$ be arbitrary

$d(x,y) = |x-y|$
$= |x-z+z-y|$
$\leq |x-z| + |z-y|$
$\leq d(x,z) + d(z,y)$

$d(x,y) \leq d(x,z) + d(z,y)$

$\therefore (R,d)$ is called m.s & d is called usual metric.

Now $f(x,z) + f(z,y) = \dfrac{d(x,z)}{1+d(x,z)} + \dfrac{d(z,y)}{1+d(z,y)}$

$$\geq \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(z,y)+d(x,z)}$$

$$= \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$f(x,z)+f(z,y) \geq \frac{1}{1+\frac{1}{d(x,z)+d(z,y)}}$$

$(\because d(x,z)+d(z,y) \neq 0)$

$$\geq \frac{1}{1+\frac{1}{d(x,y)}}$$

$$= \frac{d(x,y)}{1+d(x,y)} = f(x,y)$$

$$f(x,z)+f(z,y) \geq f(x,y)$$

or

$$f(x,y) \leq f(x,z) + f(z,y)$$

Hence M_4 is holds

$\therefore (E,f)$ is a metric space.

e.g (5) Let (E, d) be a metric spa
Let f be defined on $E \times E$
as $f(x,y) = \frac{d(x,y)}{1+d(x,y)}$, then

Prove that f is metric over E
Proof:-

Since $d(x,y) \geq 0$ &

$d(x,y) = 0$ iff $x = y$

$\therefore f(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$

and $= 0$ iff $x = y$

Hence m_1 is satisfied

For m_2

$$f(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)} = f(y,x)$$

Hence f is symmetric and f is ho

For m_3

Let $x, y, z \in E$ be arbitrary

Let us suppose $d(x,z) + d(z,y) - ($

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$T_3 = \{\phi, \{a\}, \{b\}, \{a,c\}, X\}$

Then T_1 is topology over X

(X, T_2) is not a topological space because $\{a,b\} \cap \{a,c\} = \{a\} \notin T_2$

Also (X, T_3) is not a topological space because $\{a\} \cup \{b\} = \{a,b\} \notin T_3$

Ex-5 The topology (X,T) is called

Let X be a non-empty set and let T be the family of Empty set ϕ together with all subsets of X whose complement is finite. Show that (X,T) is topological space.

Proof:-

By definition of T, $\phi \in T$ &

$$X^c = \phi \text{ (finite)}$$

$$\Rightarrow X \in T$$

Therefore 1st condition of topological space i.e., $\phi, X \in T$ holds

Now

For each $\alpha \in I$ (Index set)

let $G_\alpha \in T$

Then ~~Show~~ we have to show that

$\bigcup_{\alpha \in I} G_\alpha \in T$

We ~~cast~~ can assume that each G_α is non-empty then G_α^c is finite for each $\alpha \in I$.

Now by De Morgan's law

from β_1. But a singlet can only be the union of itself with ϕ. Hence $\{x\} \in \beta_1$

Therefore $\beta \subseteq \beta_1$.

Conversely:-

Let $\beta \subseteq \beta_1$, then β is base of $\mathcal{J}$ and hence by the previous problem β_1 is also base for $\mathcal{J}$.

✱ Sub-base :-

Let $(X, \mathcal{J})$ be a topological space. A family S of subsets of X is called (open) Sub-base for $\mathcal{J}$ (on X), if the family of all finite intersection of members of S is an open base for $\mathcal{J}$.

Ex:- Let us take a Topological space $(R, \mathcal{J})$, where $\mathcal{J}$ is usual topology on R. The family of open intervals of the form $]-\infty, b[$ and $]a, \infty[$ where a and b are real numbers is an open subbase for R. Since any open interval is either one of these or else the intersection of two of them.

for Example $]a, b[$

$$=]-\infty, b[\cap]a, \infty[$$

where $a < b$

and any open set in R is the

β^* is a topology on X
Then by theorem (B)
add the statement of theorem B
Seid are
β is an open base for this topo
and by definition of sub base, S
is an open sub base for this
topology.

Remark:-
β^* in the above topological
space is called a topology
generated by the family S $\in \mathcal{P}$
(Of all subset of X)

Theorem:- If J_1 and J_2 be any
two topologies on a set X.
Then prove that $J_1 \cap J_2$ is
also a topology on X.

Proof:- Since J_1 is a topology on X
$\Rightarrow \phi \in J_1$ and $X \in J_1$
Again, J_2 is also a topology on X
$\Rightarrow \phi \in J_2$ and $X \in J_2$
$\Rightarrow \phi \in J_1 \cap J_2$ & $X \in J_1 \cap J_2$
Therefore the 1st condition being
a topology is satisfied.
Secondly, let $G_\alpha \in J_1 \cap J_2$ for $\alpha \in D$
$\Rightarrow G_\alpha \in J_1$ and $G_\alpha \in J_2$ for

union of open intervals.
Therefore the open interval $]a, b[$ can be taken as sub-base for R.

Proposition:- Let S be a family of subsets of a given non-empty set X. Let β be the family of all finite intersection of members from S. Then β is an open base for a topology on X.
i.e., β^* is a topology over X and S is an open sub-base for this topology.

Verification :-

We first observe that $X \in \beta$ for X can be regarded as the intersection of an empty collection of members of S.
Hence $\cup\{D : D \in \beta\} = X$
Next, Let $D_1, D_2 \in \beta$, then each of D_1 and D_2 is a finite intersection of members of S, and therefore $D_1 \cap D_2 \in \beta$.
Now, for each $x \in D_1 \cap D_2$ there is a member of β namely $D_1 \cap D_2$ which contains x and is contained in $D_1 \cap D_2$.
It follows from theorem P that
(+) theorem ~~or~~ statement

is contained in E. Then by definition of open set E is an open set.

(O_2): Let $\{G_\alpha\}_{\alpha \in I}$ be a family of open sets.
Let $G = \bigcup_{\alpha \in I} G_\alpha$. Then we have to show that G is open.
For this, let $x \in G$ be arbitrary.
Then $x \in \bigcup_{\alpha \in I} G_\alpha \Rightarrow x \in G_{\alpha_0}$ for some $\alpha_0 \in I$.
Since G_{α_0} is open and $x \in G_{\alpha_0}$.
$\Rightarrow \exists$ an open sphere $S_r(x)$ with centre x s.t $S_r(x) \subseteq G_{\alpha_0}$

$$\subseteq \bigcup_{\alpha \in I} G_\alpha = G$$

Thus $S_r(x) \subseteq G$
Then by definition of open set G is open.

(O_3): It is sufficient to prove that if G and H are open sets in L, then we have to prove that $G \cap H$ is also open.
For this let $x \in G \cap H$ be arbitrary
$\Rightarrow x \in G$ and $x \in H$

Q. Let X be a non-empty set and let T be the family of empty set ϕ together with all subsets of X whose complement is countable. Then show that T is topology over X.

Co-countable topological space.